

LAGRANGIAN INCLUSION WITH AN OPEN WHITNEY UMBRELLA IS RATIONALLY CONVEX

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ABSTRACT. It is shown that a Lagrangian inclusion of a real surface in \mathbb{C}^2 with a standard open Whitney umbrella and double transverse self-intersections is rationally convex.

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1. INTRODUCTION

This paper is concerned with the study of rational convexity of compact real surfaces in \mathbb{C}^2 . A compact set X in \mathbb{C}^n is *rationally convex* if for every point p in the complement of X there exists a complex algebraic hypersurface passing through p and avoiding X . See Stout [9] for a comprehensive treatment of this fundamental notion.

A nondegenerate closed 2-form ω on \mathbb{C}^2 is called a *symplectic form*. By Darboux's theorem every symplectic form is locally equivalent to the standard form

$$\omega_{\text{st}} = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) = dd^c \phi_{\text{st}}, \quad \phi_{\text{st}} = |z|^2 + |w|^2,$$

where (z, w) , $z = x + iy$, $w = u + iv$ are complex coordinates in \mathbb{C}^2 , and $d^c = i(\bar{\partial} - \partial)$. If a symplectic form ω is of bidegree $(1, 1)$ and strictly positive, it is called a *Kähler form*. A strictly plurisubharmonic function ϕ is called a potential of ω if $dd^c \phi = \omega$. A real n -dimensional submanifold $S \subset \mathbb{C}^n$ is called *Lagrangian* for ω if $\omega|_S = 0$. According to a theorem of Duval and Sibony [2], a compact n -dimensional submanifold of \mathbb{C}^n is rationally convex if and only if it is Lagrangian for some Kähler form. This result displays a connection between rational convexity and symplectic properties of real submanifolds.

Being Lagrangian imposes certain topological restrictions on a submanifold, for example, the only compact orientable surface that admits a Lagrangian embedding into $(\mathbb{C}^2, \omega_{\text{st}})$ is a torus. On the other hand, according to the result of Givental [4], any compact surface (orientable or not) admits a *Lagrangian inclusion* into \mathbb{C}^2 , i.e., a smooth map $\iota : S \rightarrow \mathbb{C}^2$ which is a local Lagrangian embedding except a finite set of singular points that are either transverse double self-intersections or the so-called *open Whitney umbrellas*. The *standard open Whitney umbrella* is a map

$$\pi : \mathbb{R}_{(t,s)}^2 \ni (t, s) \mapsto \left(ts, \frac{2t^3}{3}, t^2, s \right) \in \mathbb{R}_{(x,u,y,v)}^4. \quad (1)$$

The open Whitney umbrella is then defined as the image of the standard umbrella under a local symplectomorphism, i.e., a local diffeomorphism that preserves the form ω_{st} . It was proved by Gayet [3] that an immersed Lagrangian (with respect to some Kähler form) submanifold in \mathbb{C}^n with transverse double self-intersections is also rationally convex. This was generalized to certain nontransverse self-intersections by Duval and Gayet [1].

The goal of this paper is show how the technique of [2], [3], and [1] can be adapted to prove rational convexity of a Lagrangian inclusion with one standard open Whitney umbrella. More precisely, we prove the following.

Theorem 1. *Let $\iota : S \mapsto (\mathbb{C}^2, \omega_{\text{st}})$ be a Lagrangian inclusion of a compact surface S . Suppose that the singularities of ι consist of transverse double self-intersections and one standard open Whitney umbrella. Then $\iota(S)$ is rationally convex in \mathbb{C}^2 .*

We remark that the standard open Whitney umbrella can be replaced by its image under a complex affine map that preserves the symplectic form ω_{st} . The existence of Lagrangian inclusions satisfying the conditions of Theorem 1 follows from a recent result of Nemirovski and Siegel [6].

2. PROOF OF THEOREM 1.

We will identify S and $\iota(S)$ as a slight abuse of notation. The ball of radius ε centred at a point p is denoted by $\mathbb{B}(p, \varepsilon)$, and the standard Euclidean distance between a point $p \in \mathbb{C}^n$ and a set $Y \subset \mathbb{C}^n$ is denoted by $\text{dist}(p, Y)$. Our approach is a modification of the method of Duval-Sibony and Gayet. The main tool here is the following result.

Lemma 2 ([2], [3]). *Let ϕ be a plurisubharmonic C^∞ -smooth function on \mathbb{C}^n , and let h be a C^∞ -smooth function on \mathbb{C}^n such that*

- (1) $|h| \leq e^\phi$, and $X := \{|h| = e^\phi\}$ is compact;
- (2) $\bar{\partial}h = O(\text{dist}(\cdot, S)^{\frac{3n+5}{2}})$;
- (3) $|h| = e^\phi$ with order 1 on S ;
- (4) *For any point $p \in X$ at least one of the following conditions hold: (i) h is holomorphic in a neighbourhood of p , or (ii) p is a smooth point of S , and ϕ is strictly plurisubharmonic at p .*

Then X is rationally convex.

The proof of Theorem 1 consists of finding the functions ϕ and h that satisfy Lemma 2 and such that the set X contains S and is contained in the union of S with the balls of arbitrarily small radius centred at singular points of S . This will be achieved in three steps: first we construct a closed $(1, 1)$ -form ω that vanishes near singular points of S and such that $\omega|_S = 0$. This is done in Section 2.1. The form ω is a modification of the standard symplectic form ω_{st} in \mathbb{C}^2 near singular points of S . Near self-intersection points this is done in the paper of Gayet [3], and so we will deal with the umbrella point. Secondly, from ω and its potential ϕ we construct the required function h . This is done in Section 2.2. In the last step, in Section 2.3, we replace ϕ with a function $\phi + \rho$, for a suitable ρ , so that the pair $\{\phi + \rho, h\}$ satisfies all the conditions of Lemma 2.

2.1. The form ω . Near the umbrella point the Lagrangian inclusion map ι coincides with π given by (1). For a function f we have

$$d^c f = -f_y dx + f_x dy - f_v du + f_u dv.$$

Direct computations show that $\pi^* d^c \phi_{\text{st}} = -2t^2 s dt - \frac{2}{3} t^3 ds$. Consider the pluriharmonic function $\zeta = \frac{v^2}{2} - \frac{u^2}{2}$. Then $\pi^* d^c \zeta = \pi^* d^c \phi_{\text{st}}$. The function

$$\phi = \phi_{\text{st}} - \zeta$$

is strictly plurisubharmonic and satisfies

$$\pi^* d^c \phi = 0. \quad (2)$$

Let $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a smooth increasing convex function such that $r(t) = 0$ when $t \leq \varepsilon_1$ and $r(t) = t - c$ when $t > \varepsilon_2$, for some suitably chosen $c > 0$ and $0 < \varepsilon_1 < \varepsilon_2$. We choose $\varepsilon_2 > 0$ so small that the set $\{\phi < \varepsilon_2\}$ does not contain singular points of S except the origin. Let

$$\omega = dd^c(r \circ \phi). \quad (3)$$

Then $\pi^* \omega = 0$ by (2). Therefore, the surface S remains Lagrangian with respect to the form ω . This gives us the required modification of ω_{st} . By construction there exist two neighbourhoods $U \Subset U'$ of the origin such that $\omega|_U = 0$ and $\omega = \omega_{\text{st}}$ in $\mathbb{C}^2 \setminus U'$, while the potential changed globally.

Denote by p_1, \dots, p_N the points of self-intersection of S , and by p_0 the standard umbrella. Then [3, Prop. 1] gives further modification $\tilde{\omega}$ of the form ω in (3), near the self-intersection points. Combining everything together yields the following result.

Lemma 3. *Given $\varepsilon > 0$ sufficiently small, there exists a $(1,1)$ -form $\tilde{\omega}$ and $\varepsilon_1 > 0$, such that*

- (i) $\tilde{\omega}|_S = 0$;
- (ii) $\tilde{\omega} = \omega$ on $\mathbb{C}^2 \setminus \cup_{j=0}^N \mathbb{B}(p_j, \varepsilon)$.
- (iii) $\tilde{\omega}$ vanishes on $\mathbb{B}(p_j, \varepsilon_1)$, $j = 0, \dots, N$.

Furthermore, there exists a smooth function $\tilde{\phi}$ on \mathbb{C}^2 such that $dd^c \tilde{\phi} = \tilde{\omega}$. The function $\tilde{\phi}$ is plurisubharmonic on \mathbb{C}^2 , and strictly plurisubharmonic on $\mathbb{C}^2 \setminus \cup_{j=0}^N \mathbb{B}(p_j, \varepsilon)$.

2.2. The function h . Let $\iota : S \rightarrow \mathbb{C}^2$ be a Lagrangian inclusion, and $\tilde{\phi}$ be the potential of the form $\tilde{\omega}$ given by Lemma 3. For simplicity we drop tilde from the notation. In this subsection we recall the construction in [2] and [3] of a smooth function h on \mathbb{C}^2 such that $|h| |_S = e^\phi$ and $\bar{\partial}h(z) = O(\text{dist}(z, S)^6)$. The two conditions, that $\bar{\partial}h$ vanishes on S and that $\phi - \log |h|$ vanishes on S with order 1 imply that $\iota^*(d^c \phi - d(\arg h)) = 0$. The latter condition can be met by further perturbation of ϕ .

Let \tilde{S} be the deformation retract of S . Note that it exists because near the umbrella point the surface S is the graph of a continuous vector-function. Let γ_k , $k = 1, \dots, m$, be the basis in $H_1(\tilde{S}, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$ supported on S . Using de Rham's theorem and an argument similar to that of Lemma 3 one can find smooth functions ψ_k with compact support in \mathbb{C}^2 that vanish on $S \cup (\cup_j B(p_j, \varepsilon))$, where $B(p_j, \varepsilon)$ are the balls around the singular points on S as in Lemma 3, such that $\int_{\gamma_k} \iota^* d^c \psi_l = \delta_{kl}$. Further, one can find small rational numbers λ_k and an integer M , such that for the function

$$\tilde{\phi} = M \left(\phi + \sum_{j=1}^m \lambda_k \psi_k \right) \quad (4)$$

the form $\iota^* d^c \tilde{\phi}$ is closed on S and has periods which are multiples of 2π . Then there exists a C^∞ -smooth function $\mu : S \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ that vanishes on the intersection of S with $B(p_j, \varepsilon)$, $j = 0, \dots, N$, and such that $\iota^* d^c \tilde{\phi} = d\mu$. By [5], there exists a function h defined on \mathbb{C}^2 such that

$$h|_S = e^{\tilde{\phi} + i\mu}|_S$$

and $\bar{\partial}h(z) = O(\text{dist}(z, S)^6)$. It follows that $\tilde{\phi} - \log|h|$ vanishes to order 1 on S . Note that h is constant near singular points of S .

2.3. The function ϕ . Again, for simplicity of notation we denote by ϕ the function (4) constructed in Section 2.2. It does not yet satisfy the conditions of Lemma 2 because there are still some smooth points on S where the function h is not holomorphic and ϕ is not strictly plurisubharmonic. For this we will replace ϕ by a function $\tilde{\phi} = \phi + c \cdot \rho$, where the function ρ will be constructed using local polynomial convexity of S , and $c > 0$ will be a suitable constant.

We recall our result from [7, 8].

Lemma 4. *Let S be a Lagrangian inclusion in \mathbb{C}^2 , and let p_0, \dots, p_N be its singular points. Suppose that S is locally polynomially convex near every singular point. Then there exists a neighbourhood Ω of S in \mathbb{C}^2 and a continuous non-negative plurisubharmonic function ρ on Ω such that $S \cap \Omega = \{p \in \Omega : \rho(p) = 0\}$. Furthermore, for every $\delta > 0$ one can choose $\rho = (\text{dist}(z, S))^2$ on $\Omega \setminus \bigcup_{j=0}^N \mathbb{B}(p_j, \delta)$; in particular, it is smooth and strictly plurisubharmonic there.*

The standard open Whitney umbrella is locally polynomially convex by [7], and S is locally polynomially convex near transverse double self-intersection points by [8]. For the proof of the lemma we refer the reader to [8].

To complete the construction of the function ϕ , we choose the function ρ in Lemma 4 with $\delta > 0$ so small that the balls $\mathbb{B}(p_j, \delta)$ are contained in balls $\mathbb{B}(p_j, \varepsilon_1/2)$ given by Lemma 3. Note that ρ is defined only in a neighbourhood Ω of S , but we can extend it as a smooth function with compact support in \mathbb{C}^2 . Consider now the function

$$\tilde{\phi} = \phi + c \cdot \rho.$$

We choose the constant $c > 0$ so small that the function $\tilde{\phi}$ remains to be plurisubharmonic on \mathbb{C}^2 . At the same time, since $c > 0$ and ρ is strictly plurisubharmonic on S outside small neighbourhoods of singular points, we conclude that the function $\tilde{\phi}$ is strictly plurisubharmonic outside the balls $\mathbb{B}(p_j, \delta)$.

The pair $\tilde{\phi}$ and h now satisfies all the conditions of Lemma 2. This completes the proof of Theorem 1.

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